

Permanental polynomials of skew adjacency matrices of oriented graphs

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Abstract

Let G^σ be an orientation of a simple graph G and $A_s(G^\sigma)$ the skew adjacency matrix of G^σ . In this paper, the permanental polynomial of $A_s(G^\sigma)$ is investigated. All the coefficients of the permanental polynomial of $A_s(G^\sigma)$ are presented in terms of G^σ , and it is proved that the skew adjacency matrices of a graph G all have the same permanental polynomial if and only if G has no even cycles. Moreover, the relationship between $S_p(G^\sigma)$ and $S_p(G)$ is studied, where $S_p(G^\sigma)$ and $S_p(G)$ denote respectively the per-spectrum of G^σ and G . In particular, (i) $S_p(G^\sigma) = iS_p(G)$ for some orientation G^σ if and only if G is bipartite, (ii) $S_p(G^\sigma) = iS_p(G)$ for any orientation G^σ if and only if G is a forest, where $i^2 = -1$.

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1 Introduction

Let G be a simple graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $\{v_i, v_j\} \in E(G)$ and $a_{ij} = 0$ otherwise. An oriented graph G^σ is a simple graph G with an orientation σ , which assigns to each edge a direction

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so that G^σ becomes a directed graph. Both σ and G^σ are called orientations of G . The *skew adjacency matrix* of G^σ , denoted by $A_s(G^\sigma)$, is defined to be the $n \times n$ matrix (a_{ij}) whose (i, j) -entry a_{ij} satisfies

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E(G^\sigma), \\ -1, & \text{if } (v_j, v_i) \in E(G^\sigma), \\ 0, & \text{otherwise.} \end{cases}$$

It is known that the skew adjacency matrix plays an important role in enumeration of perfect matchings, since the square of the number of perfect matchings of a graph G with a Pfaffian orientation G^σ is equal to the determinant of $A_s(G^\sigma)$ (see [19] for more details). It is worth mentioning that Tutte's original proof [22] of his famous perfect matching theorem made use of the following criterion: A graph G has a perfect matching if and only if $\det B(\mathbf{x})$ is not identically 0, where $B(\mathbf{x}) = (b_{ij})$ is a skew symmetric matrix associated to an orientation G^σ of G , whose entry $b_{ij} = x_e$ if $e = (v_i, v_j)$, $b_{ij} = -x_e$ if $e = (v_j, v_i)$, $b_{ij} = 0$ if v_i and v_j are non-adjacent, and x_e is a variable assigned to arc e of G^σ . Clearly, $B(\mathbf{x})$ is precisely the skew adjacency matrix $A_s(G^\sigma)$ of G^σ if each x_e is equal to 1.

Recently, the skew energy of an oriented graph G^σ was introduced by Adiga et al. [1], which is defined as the sum of the absolute values of the eigenvalues of $A_s(G^\sigma)$. In [1], some open problems were posed, and one of them was to interpret all the coefficients of the characteristic polynomial of $A_s(G^\sigma)$ in terms of G^σ . This problem was resolved independently by several authors (see [9], [14] and [16]). Cavers et al. [9] proved that the skew adjacency matrices of a graph G are all cospectral if and only if G has no even cycles, and investigated the maximum value $\rho_s(G)$ of the spectral radii of the skew adjacency matrices of a graph G . In [21], the relationship between $S(G^\sigma)$ and $S(G)$ was studied, where $S(G^\sigma)$ and $S(G)$ denote respectively the spectrum of $A_s(G^\sigma)$ and $A(G)$.

In this paper, we are interested in the permanental polynomials of skew adjacency matrices of oriented graphs.

The *permanent* of an $n \times n$ matrix M with entries m_{ij} ($i, j = 1, 2, \dots, n$) is defined by

$$\text{per} M = \sum_{\pi \in S_n} \prod_{i=1}^n m_{i\pi(i)},$$

where S_n is the symmetric group on n elements. While the determinant can be calculated using Gaussian elimination, no efficient algorithm for computing the permanent is known. In [23], Valiant has shown that computing the permanent is #P-complete even when restricted to (0,1)-matrices.

Let G be a graph on n vertices and $A(G)$ the adjacency matrix of G . The *permanental polynomial* of G , $\pi(G, x)$, is defined by

$$\pi(G, x) = \text{per}(xI - A(G)), \quad (1)$$

where I is the $n \times n$ identity matrix.

Analogously, the *permanental polynomial* of G^σ , $\pi(G^\sigma, x)$, is defined by

$$\pi(G^\sigma, x) = \text{per}(xI - A_s(G^\sigma)). \quad (2)$$

We can write (1) and (2) in the coefficient form

$$\pi(G, x) = \sum_{k=0}^n a_k(G) x^{n-k},$$

and

$$\pi(G^\sigma, x) = \sum_{k=0}^n a_k(G^\sigma) x^{n-k}.$$

In what follows, we abbreviate the coefficients $a_k(G)$ and $a_k(G^\sigma)$ to a_k when no confusion can arise.

The roots of the permanental polynomial of G (resp. G^σ) are called the *permanental roots* of G (resp. G^σ). Borowiecki [3] defined the *per-spectrum* $S_p(G)$ of G as the multiset of permanental roots of G . Analogously, the *per-spectrum* $S_p(G^\sigma)$ of G^σ is defined as the multiset of permanental roots of G^σ .

The permanental polynomial of a graph was first systematically studied by Merris et al. [20], and the study of this polynomial in chemical literature was started by Kasum et al. [17]. Since it is hard to be computed, the literature on permanental polynomials is far less than that on characteristic polynomials of graphs (we refer the reader to [2–8, 10, 15, 17, 18, 20, 24–26]). It is worth pointing out that Yan and Zhang [24] proved that for a bipartite graph G containing no even subdivisions of $K_{2,3}$, there exists an orientation G^σ of G such that $\pi(G, x) = \det(xI - A_s(G^\sigma))$.

The rest of the paper is organized as follows. Section 2 reviews some known results which relate the coefficients of the permanental polynomial of a graph with structural properties of the given graph. In Section 3, we obtain the coefficient formula of $\pi(G^\sigma, x)$, which expresses the coefficient a_k of x^{n-k} in the permanental polynomial $\pi(G^\sigma, x)$ in terms of collections of vertex disjoint edges and even cycles of G that cover k vertices, and we generalize this formula to weighted oriented graphs, which provides a graph-theoretic method to compute the permanent of real skew symmetric matrices. In Section 4, we show that the skew adjacency matrices of a graph G all have the same permanental polynomial if and only if G has no even cycles. It is also shown that a graph G has no even cycles if and only if the permanental polynomial of $A_s(G^\sigma)$ equals the matching polynomial of G for all orientations G^σ . Furthermore, the relationship between $S_p(G^\sigma)$ and $S_p(G)$ is investigated.

2 Permanental polynomials of adjacency matrices

In this section, we present some known results on the coefficients of the permanental polynomials of (directed) multigraphs (loops are allowed), and weighted (directed) graphs, respectively.

The adjacency matrix $A(G) = (a_{ij})$ of a directed multigraph G with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ is an $n \times n$ matrix, whose entry a_{ij} is equal to the number of arcs starting at the vertex v_i and terminating at the vertex v_j . For an undirected multigraph G , the entry a_{ij} of the adjacency matrix $A(G)$ of G is the number of edges between v_i and v_j . A *linear directed graph* is a directed graph in which each vertex has indegree one and outdegree one, i.e., it consists of directed cycles.

Theorem 2.1 ([11]). *Let G be a directed multigraph and $A(G)$ the adjacency matrix of G . If $\pi(G, x) = \text{per}(xI - A(G)) = \sum_{k=0}^n a_k x^{n-k}$, then*

$$a_k = (-1)^k \sum_{L \in \mathcal{L}_k} 1, \quad 1 \leq k \leq n,$$

where \mathcal{L}_k is the set of all linear directed subgraphs of G with exactly k vertices.

A *Sachs graph* (also called *basic figure* [11]; *elementary graph* [12]) is an undirected graph in which each component is a single edge or a cycle C_p with p ($p \geq 1$) vertices (loops being included with $p = 1$).

Theorem 2.2 ([11]). *Let G be an undirected multigraph and $A(G)$ the adjacency matrix of G . If $\pi(G, x) = \text{per}(xI - A(G)) = \sum_{k=0}^n a_k x^{n-k}$, then*

$$a_k = (-1)^k \sum_{U \in \mathcal{U}_k} 2^{c(U)}, \quad 1 \leq k \leq n,$$

where \mathcal{U}_k is the set of all Sachs subgraphs of G with exactly k vertices, and $c(U)$ is the number of cycles in U .

Let us point out that Theorems 2.1 and 2.2 are also obtained in [4], and the coefficients of the permanental polynomial of a simple graph are obtained independently by several authors (see, for example, [17] and [20]).

Theorem 2.3. *Let G be a simple graph and $A(G)$ the adjacency matrix of G . If $\pi(G, x) = \text{per}(xI - A(G)) = \sum_{k=0}^n a_k x^{n-k}$, then*

$$a_k = (-1)^k \sum_{U \in \mathcal{U}_k} 2^{c(U)}, \quad 1 \leq k \leq n.$$

Theorems 2.1 and 2.2 can be extended to weighted (directed) graphs. We may assume that there is at most one arc from v_i to v_j , and that a_{ij} is the weight of this arc. Let $A(G) = (a_{ij})$ be the corresponding generalized adjacency matrix.

Theorem 2.4 ([11]). *Let G be a weighted directed graph and $A(G)$ the generalized adjacency matrix of G . If $\pi(G, x) = \text{per}(xI - A(G)) = \sum_{k=0}^n a_k x^{n-k}$, then*

$$a_k = (-1)^k \sum_{L \in \mathcal{L}_k} \prod(L), \quad 1 \leq k \leq n,$$

where $\prod(L)$ denotes the product of the weights of all arcs belonging to L .

If G is a weighted undirected graph and U is a Sachs subgraph of G , let

$$\prod(U) = \prod_{e \in E(U)} (w(e))^{\zeta(e; U)},$$

where $E(U)$ is the set of edges of U , $w(e)$ is the weight of e , and

$$\zeta(e; U) = \begin{cases} 1, & \text{if } e \text{ is contained in some cycle of } U, \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 2.5 ([11]). *Let G be a weighted undirected graph and $A(G)$ the generalized adjacency matrix of G . If $\pi(G, x) = \text{per}(xI - A(G)) = \sum_{k=0}^n a_k x^{n-k}$, then*

$$a_k = (-1)^k \sum_{U \in \mathcal{U}_k} 2^{c(U)} \prod(U), \quad 1 \leq k \leq n.$$

If we consider an arbitrary $n \times n$ matrix $A = (a_{ij})$ as the generalized adjacency matrix of a weighted directed graph G , then by Theorem 2.4, we have

$$\text{per} A = \sum_{L \in \mathcal{L}_n} \prod(L),$$

where \mathcal{L}_n is the set of all spanning linear directed subgraphs of G .

Remark 2.6. In fact, Cvetković et al. [11, pp. 34–35] obtained the coefficients of the perm-polynomials of the adjacency matrices of (directed) multigraphs, and weighted (directed) graphs, where the *perm-polynomial* of a square matrix A of order n is defined as

$$\text{per}(xI + A) = x^n + a_1^* x^{n-1} + \cdots + a_n^*.$$

Clearly, if $\text{per}(xI - A) = x^n + a_1 x^{n-1} + \cdots + a_n$, then $a_i = (-1)^i a_i^*$ for $1 \leq i \leq n$. Thus we immediately obtain Theorems 2.1–2.5.

3 Coefficients of the permanental polynomials of skew adjacency matrices

In this section, we obtain the coefficient formula of the permanental polynomial of an oriented graph, and generalize this result to weighted oriented graph.

Recall that S_n is the symmetric group on n elements. It is well-known that every permutation in S_n is a product of disjoint cycles. Denote by $\mathcal{E}(n)$ the set of all permutations in S_n with all cycles having even length.

Lemma 3.1. *Let $A = (a_{ij})$ be an $n \times n$ skew symmetric matrix. Then*

$$\text{per} A = \sum_{\pi \in \mathcal{E}(n)} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

Proof. Let $\pi = r_1 r_2 \dots r_t$ be a permutation in $S_n \setminus \mathcal{E}(n)$. Since A is a skew symmetric matrix, $a_{ii} = 0$ for $i = 1, 2, \dots, n$. Thus π will contribute 0 to $\text{per} A$ if π has a fixed point. So we assume that π is fixed-point-free. We define the least element of a cycle r_i of π to be the least element of $\{1, 2, \dots, n\}$ in r_i . Since $\pi \in S_n \setminus \mathcal{E}(n)$, π contains an odd cycle. We obtain π' from π by only reversing the odd cycle with the smallest least element. It is easy to see that $\pi'' = \pi$ and $\prod_{i=1}^n a_{i\pi(i)} = -\prod_{i=1}^n a_{i\pi'(i)}$. Thus we have partitioned the fixed-point-free permutations in $S_n \setminus \mathcal{E}(n)$ into pairs such that each pair contributes 0 to $\text{per} A$. This completes the proof. \square

Let G^σ be an orientation of a graph G and C an even cycle in G . We say that C is *oddly* (resp. *evenly*) *oriented* if for either choice of direction of traversal around C , the number of oriented edges of C whose orientation agrees with the direction of traversal is odd (resp. even).

Theorem 3.2. *Let G^σ be an orientation of a graph G and $A_s(G^\sigma)$ the skew adjacency matrix of G^σ . If $\pi(G^\sigma, x) = \sum_{k=0}^n a_k x^{n-k}$, then*

$$a_k = \sum_{U \in \mathcal{U}_k} (-1)^{m(U) + c^-(U)} 2^{c(U)},$$

where \mathcal{U}_k is the set of all Sachs subgraphs of G on k vertices with no odd cycles, $m(U)$ is the number of single edges of U and $c^-(U)$ is the number of oddly oriented cycles of U relative to G^σ . In particular, $a_k = 0$ if k is odd.

Proof. Let us first consider the permanent of $A_s(G^\sigma)$. By Lemma 3.1, we have

$$\text{per}(A_s(G^\sigma)) = \sum_{\pi \in \mathcal{E}(n)} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

Let $\pi = r_1 r_2 \dots r_t$ be a permutation in $\mathcal{E}(n)$. The term $a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}$ is nonzero if and only if $a_{i\pi(i)} \neq 0$ for $i = 1, 2, \dots, n$, i.e., $(v_i, v_{\pi(i)})$ or $(v_{\pi(i)}, v_i)$ is an arc of G^σ . If $a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} \neq 0$, then this term determines a Sachs subgraph $U \in \mathcal{U}_n$ in which the components isomorphic to the complete graph K_2 are determined by the transpositions among the r_i , and the even cycles are determined by the remaining r_i . Conversely, U arises from $2^{c(U)}$ permutations, namely $r_1^{\pm 1} r_2^{\pm 1} \cdots r_{c(U)}^{\pm 1} r_{c(U)+1} \cdots r_t$, where $r_1, r_2, \dots, r_{c(U)}$ are the r_i of length greater than 2. It is easy to see that a

single edge contributes -1 , an oddly oriented cycle contributes -1 , and an evenly oriented cycle contributes 1 to the term $a_{1\pi(1)}a_{2\pi(2)}\dots a_{n\pi(n)}$, respectively. Thus we have

$$\text{per}(A_s(G^\sigma)) = \sum_{U \in \mathcal{E}\mathcal{U}_n} (-1)^{m(U)+c^-(U)} 2^{c(U)}.$$

If $n = |V(G)|$ is odd, then $\mathcal{E}(n) = \emptyset$, and consequently $\text{per}(A_s(G^\sigma)) = 0$.

It is known that $(-1)^k a_k$ equals the sum of all $k \times k$ principal subpermanents of $A_s(G^\sigma)$. Note that there is a one-to-one correspondence between the set of these principal subpermanents and the set of all induced subgraphs of G having exactly k vertices. Applying the result obtained above to each of the $\binom{n}{k}$ principal subpermanents and summing, we complete the proof. \square

Theorem 3.2 can be extended to weighted oriented graphs. We first present the following definition. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. Let G_ω^σ be an oriented graph G^σ with a weight function ω , which assigns to each arc (v_i, v_j) a weight ω_{ij} so that G_ω^σ becomes a weighted oriented graph. The generalized skew adjacency matrix of G_ω^σ is defined to be the $n \times n$ matrix $A_s(G_\omega^\sigma) = (a_{ij})$, whose (i, j) -entry a_{ij} satisfies

$$a_{ij} = \begin{cases} \omega_{ij}, & \text{if } (v_i, v_j) \in E(G_\omega^\sigma), \\ -\omega_{ji}, & \text{if } (v_j, v_i) \in E(G_\omega^\sigma), \\ 0, & \text{otherwise.} \end{cases}$$

Following similar arguments as in the proof of Theorem 3.2, we can immediately obtain the coefficients of the permanent polynomial of a weighted oriented graph.

Theorem 3.3. *Let G_ω^σ be a weighted oriented graph of a graph G and $A_s(G_\omega^\sigma)$ the generalized skew adjacency matrix of G_ω^σ . If $\pi(G_\omega^\sigma, x) = \text{per}(xI - A_s(G_\omega^\sigma)) = \sum_{k=0}^n a_k x^{n-k}$, then $a_k = 0$ if k is odd, and*

$$a_k = \sum_{U \in \mathcal{E}\mathcal{U}_k} (-1)^{m(U)+c^-(U)} 2^{c(U)} \prod(U), \quad \text{if } k \text{ is even,} \quad (3)$$

where $\prod(U)$ is defined as in Theorem 2.5. In particular, we have

$$\text{per}(A_s(G_\omega^\sigma)) = \sum_{U \in \mathcal{E}\mathcal{U}_n} (-1)^{m(U)+c^-(U)} 2^{c(U)} \prod(U). \quad (4)$$

Suppose that $A = (a_{ij})$ is an $n \times n$ real skew symmetric matrix. We can construct a weighted oriented graph G_ω^σ associated to A as follows: let $\{v_1, v_2, \dots, v_n\}$ be the vertex set of G_ω^σ such that v_i corresponds with the i -th row (and the corresponding column) of A , $(v_i, v_j) \in E(G_\omega^\sigma)$ if and only if

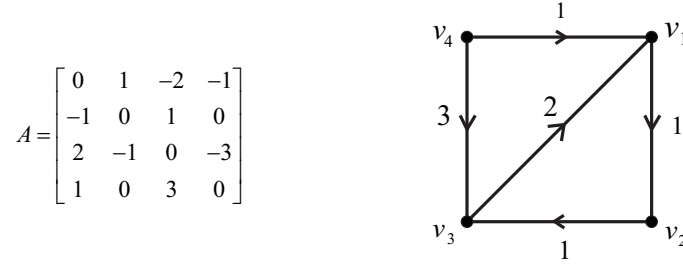


Figure 1: A skew symmetric matrix A and its associated weighted oriented graph G_w^σ .

$a_{ij} > 0$, and we assign weight a_{ij} to the arc (v_i, v_j) . An example is illustrated in Fig. 1.

Clearly, the weighted oriented graph G_w^σ associated to A has the generalized skew adjacency matrix equal to A . Therefore, by Theorem 3.3, we can calculate the permanental polynomial of an arbitrary real skew symmetric matrix A considered as the generalized skew adjacency matrix of a weighted oriented graph G_w^σ (associated to A). For example, applying (3) and (4) to the weighted oriented graph G_w^σ associated to the 4×4 matrix A above, we obtain $\text{per}(xI_4 - A) = x^4 - 16x^2 + 4$ and $\text{per}A = 4$.

4 Permanental roots of skew adjacency matrices

Let G be a simple graph with m edges. Since each edge has two possible directions, it follows that G has 2^m distinct orientations. It is of interest to know whether all the skew adjacency matrices of G can have the same permanental polynomial (or equivalently the same per-spectrum). The next theorem gives us an affirmative answer to this question.

An r -*matching* in a graph G is a set of r edges, no two of which have a vertex in common. The number of r -matchings in G will be denoted by $p(G, r)$.

Theorem 4.1. *All orientations G^σ of a graph G have the same permanental polynomial if and only if G has no even cycles.*

Proof. The sufficiency can be easily seen from Theorem 3.2. Now we are going to prove the necessity of this theorem by contradiction. We assume that all orientations G^σ have the same permanental polynomial, and G contains an even cycle.

Let $2l$ be the smallest length of an even cycle in G . It is easy to see that the Sachs subgraphs of $\mathcal{E}\mathcal{U}_{2l}$ are l -matchings or $2l$ -cycles. Let G^σ be

an orientation of G . By Theorem 3.2, we have

$$a_{2l} = (-1)^l p(G, l) + 2 \sum_{C \in \mathcal{C}} (-1)^{c^-(C)}, \quad (5)$$

where \mathcal{C} is the set of all $2l$ -cycles in G .

For an edge e , denote by $n_+(e)$ (resp. $n_-(e)$) the number of evenly (resp. oddly) oriented $2l$ -cycles in G containing e . We claim that $n_+(e) = n_-(e)$. Suppose, to the contrary, that $n_+(e) \neq n_-(e)$. Consider the new orientation of G obtained from G^σ by only reversing the orientation of e . Then in Eq. (5) the contribution from the l -matchings and those $2l$ -cycles not containing e will be unaffected, whereas the contribution from $2l$ -cycles containing e equals $2(n_+(e) - n_-(e))$ and will be negated. It follows that a_{2l} will change under this new orientation of G , which contradicts all orientations G^σ have the same permanent polynomial.

For $t \in \{1, 2, \dots, 2l\}$, denote by $n_+(e_1, e_2, \dots, e_t)$ (resp. $n_-(e_1, e_2, \dots, e_t)$) the number of evenly (resp. oddly) oriented $2l$ -cycles in G containing all of e_1, e_2, \dots, e_t .

We claim that for each $t \in \{1, 2, \dots, 2l\}$, $n_+(e_1, e_2, \dots, e_t) = n_-(e_1, e_2, \dots, e_t)$ for all orientations G^σ and all edges e_1, e_2, \dots, e_t . We proceed by induction on t .

The case $t = 1$ has been proved as above. Assume that the claim holds for $t < 2l$. Let G^σ be an orientation of G . For edges $e_1, e_2, \dots, e_t, e_{t+1}$ of G , denote by $n_+(e_1, e_2, \dots, e_t, \overline{e_{t+1}})$ (resp. $n_-(e_1, e_2, \dots, e_t, \overline{e_{t+1}})$) the number of evenly (resp. oddly) oriented $2l$ -cycles in G containing e_1, e_2, \dots, e_t , but not e_{t+1} . It is easy to see that

$$n_+(e_1, e_2, \dots, e_t) = n_+(e_1, e_2, \dots, e_t, e_{t+1}) + n_+(e_1, e_2, \dots, e_t, \overline{e_{t+1}}),$$

and

$$n_-(e_1, e_2, \dots, e_t) = n_-(e_1, e_2, \dots, e_t, e_{t+1}) + n_-(e_1, e_2, \dots, e_t, \overline{e_{t+1}}).$$

Consider the orientation $G^{\tilde{\sigma}}$ obtained from G^σ by only reversing the orientation of e_{t+1} . Then

$$\tilde{n}_+(e_1, e_2, \dots, e_t) = n_-(e_1, e_2, \dots, e_t, e_{t+1}) + n_+(e_1, e_2, \dots, e_t, \overline{e_{t+1}}),$$

and

$$\tilde{n}_-(e_1, e_2, \dots, e_t) = n_+(e_1, e_2, \dots, e_t, e_{t+1}) + n_-(e_1, e_2, \dots, e_t, \overline{e_{t+1}}).$$

By the induction hypothesis, we have

$$n_+(e_1, e_2, \dots, e_t) = n_-(e_1, e_2, \dots, e_t),$$

and

$$\tilde{n}_+(e_1, e_2, \dots, e_t) = \tilde{n}_-(e_1, e_2, \dots, e_t).$$

Thus, we have

$$\begin{aligned}
& n_+(e_1, e_2, \dots, e_t, \overline{e_{t+1}}) - n_-(e_1, e_2, \dots, e_t, \overline{e_{t+1}}) \\
&= n_-(e_1, e_2, \dots, e_t, e_{t+1}) - n_+(e_1, e_2, \dots, e_t, e_{t+1}) \\
&= n_+(e_1, e_2, \dots, e_t, e_{t+1}) - n_-(e_1, e_2, \dots, e_t, e_{t+1}).
\end{aligned}$$

This gives $n_+(e_1, e_2, \dots, e_t, e_{t+1}) = n_-(e_1, e_2, \dots, e_t, e_{t+1})$. The claim holds from the principle of induction.

Let C be an even cycle of length $2l$ and e_1, e_2, \dots, e_{2l} the edges of C . By the above claim, we have $n_+(e_1, e_2, \dots, e_{2l}) = n_-(e_1, e_2, \dots, e_{2l})$ for any orientation G^σ . This is impossible, since one side of the equality is 1, and the other is 0. \square

The argument of the proof of Theorem 4.1 is quite similar to the one used to prove Theorem 4.2 in [9], which states that the skew adjacency matrices of a graph G are all cospectral if and only if G has no even cycles. It is easy to see that if G has no even cycles, then each block of G is either a complete graph K_2 on two vertices or an odd cycle.

The *matching polynomial* [13] of a graph G on n vertices is defined by

$$\mu(G, x) = \sum_{r \geq 0} (-1)^r p(G, r) x^{n-2r}.$$

By Theorems 3.2 and 4.1, we immediately obtain the following interesting result.

Corollary 4.2. *A graph G has no even cycles if and only if $\pi(G^\sigma, x) = \mu(G, x)$ for any orientation G^σ of G .*

Remark 4.3. In [26], the authors showed that if a graph G contains at least one edge then G must have complex permanental roots. It is well-known that the roots of the matching polynomial of a graph are all real [13]. By Corollary 4.2, the permanental roots of any orientation G^σ of a graph G having no even cycles are all real.

Recall that $S_p(G^\sigma)$ and $S_p(G)$ are the per-spectrum of G^σ and G , respectively. Now we are in a position to investigate the relationship between $S_p(G^\sigma)$ and $S_p(G)$. First, it is easy to see that $a_2(G^\sigma) = -|E(G)|$ and $a_2(G) = |E(G)|$ from Theorems 3.2 and 2.3, where $a_2(G^\sigma)$ and $a_2(G)$ are the coefficients of x^{n-2} of $\pi(G^\sigma, x)$ and $\pi(G, x)$, respectively. It follows that $\pi(G^\sigma, x) = \pi(G, x)$ for some orientation G^σ of G if and only if G is empty (i.e. G has no edges). Equivalently, $S_p(G^\sigma) = S_p(G)$ for some orientation G^σ of G if and only if G is empty.

Lemma 4.4 ([5]). *Let G be a graph on n vertices with $\pi(G, x) = \sum_{k=0}^n a_k x^{n-k}$. Then G is bipartite if and only if $a_k = 0$ for all odd k .*

The following result is easy to verify and the proof is omitted.

Lemma 4.5. Suppose $\pi(G, x) = \sum_{k=0}^n a_k(G)x^{n-k}$ and $\pi(G^\sigma, x) = \sum_{k=0}^n a_k(G^\sigma)x^{n-k}$. Then $S_p(G^\sigma) = iS_p(G)$ if and only if $a_k(G^\sigma) = a_k(G) = 0$ if k is odd, $a_k(G^\sigma) = a_k(G)$ if $k \equiv 0 \pmod{4}$, and $a_k(G^\sigma) = -a_k(G)$ if $k \equiv 2 \pmod{4}$, where $i^2 = -1$.

Theorem 4.6. A graph G is bipartite if and only if there exists an orientation G^σ such that $S_p(G^\sigma) = iS_p(G)$, where $i^2 = -1$.

Proof. Suppose that $\pi(G, x) = \sum_{k=0}^n a_k(G)x^{n-k}$ and $\pi(G^\sigma, x) = \sum_{k=0}^n a_k(G^\sigma)x^{n-k}$. If there exists an orientation G^σ of G such that $S_p(G^\sigma) = iS_p(G)$, then from Lemma 4.5, $a_k(G) = a_k(G^\sigma) = 0$ for all odd k . By Lemma 4.4, G is bipartite.

Conversely, we assume that G is bipartite and (X, Y) is a bipartition of G . Orient G by directing all edges of G toward Y . Denote this orientation by G^σ . Let C_{2l} be an even cycle in G of length $2l$. Then C_{2l} is oddly oriented if and only if l is odd.

By Theorems 2.3 and 3.2, we have

$$a_{2l}(G) = \sum_{U \in \mathcal{U}_{2l}} 2^{c(U)} = p(G, l) + \sum_{\substack{U \in \mathcal{U}_{2l} \\ c(U) > 0}} 2^{c(U)},$$

and

$$a_{2l}(G^\sigma) = (-1)^l p(G, l) + \sum_{\substack{U \in \mathcal{E}\mathcal{U}_{2l} \\ c(U) > 0}} (-1)^{m(U) + c^-(U)} 2^{c(U)}.$$

Since G is bipartite, it follows that $\mathcal{U}_{2l} = \mathcal{E}\mathcal{U}_{2l}$. Let U be a Sachs subgraph of G on $2l$ vertices containing at least one cycle. Let $H_1, H_2, \dots, H_{c(U)}$ be all the cycles of U . Without loss of generality, we assume that $H_1, H_2, \dots, H_{c^-(U)}$ are oddly oriented relative to G^σ . Suppose that $l(H_j) = 2(2s_j + 1)$ for $j = 1, 2, \dots, c^-(U)$ and $l(H_j) = 2(2s_j)$ for $j = c^-(U) + 1, \dots, c(U)$, where $l(H_j)$ denotes the length of cycle H_j . Thus

$$2m(U) + 2 \sum_{j=1}^{c^-(U)} (2s_j + 1) + 2 \sum_{j=c^-(U)+1}^{c(U)} (2s_j) = 2l.$$

It follows that $m(U) + c^-(U) \equiv l \pmod{2}$. Therefore,

$$a_{2l}(G^\sigma) = (-1)^l p(G, l) + \sum_{\substack{U \in \mathcal{E}\mathcal{U}_{2l} \\ c(U) > 0}} (-1)^l 2^{c(U)} = (-1)^l a_{2l}(G).$$

Clearly, if G contains no cycles, then $a_{2l}(G^\sigma) = (-1)^l p(G, l) = (-1)^l a_{2l}(G)$. Since G is bipartite, we have $a_{2l+1}(G) = 0$ from Lemma 4.4. By Theorem 3.2, we have $a_{2l+1}(G^\sigma) = 0$. Thus $a_{2l+1}(G^\sigma) = a_{2l+1}(G) = 0$. Therefore $S_p(G^\sigma) = iS_p(G)$ from Lemma 4.5. \square

Theorem 4.7. *For any orientation G^σ of G , $S_p(G^\sigma) = iS_p(G)$ if and only if G is a forest, where $i^2 = -1$.*

Proof. Suppose that $S_p(G^\sigma) = iS_p(G)$ for any orientation G^σ . Then G has no even cycles by Theorem 4.1. It follows from Theorem 4.6 that G is bipartite. Thus G is a forest.

Conversely, we assume that G is a forest. By Theorems 4.1 and 4.6, we have $S_p(G^\sigma) = iS_p(G)$ for any orientations G^σ . \square

Recall that $S(G)$ is the adjacency spectrum of G . Borowiecki [3] showed that $S_p(G) = iS(G)$ if and only if G is a bipartite graph without cycles of length $4l$ ($l \geq 1$), where $i^2 = -1$. Thus, by Theorems 4.6 and 4.7, we immediately obtain the following corollaries.

Corollary 4.8. *If G is a bipartite graph without cycles of length $4l$ ($l \geq 1$), then there exists an orientation G^σ of G such that $S_p(G^\sigma) = S(G)$.*

Proof. Let G be a bipartite graph without cycles of length $4l$ ($l \geq 1$). Then by Theorem 4.6 and Borowiecki's result, there exists an orientation G^σ such that $S_p(G^\sigma) = -S(G)$. Since the spectrum of a bipartite graph is symmetric with respect to the origin [11], we have $-S(G) = S(G)$. Thus $S_p(G^\sigma) = S(G)$. \square

The converse of Corollary 4.8 is not true. For example, we consider the cycle C_4 of length 4. It is easy to see that $S(C_4) = \{-2, 0, 0, 2\}$. If C_4 is oddly oriented, then by Theorem 3.2, we have $\pi(C_4^\sigma, x) = x^4 - 4x^2$ and $S_p(C_4^\sigma) = \{-2, 0, 0, 2\}$.

Corollary 4.9. *For any orientation G^σ of G , $S_p(G^\sigma) = S(G)$ if and only if G is a forest.*

Proof. If G is a forest, then by Theorem 4.7 and Borowiecki's result, $S_p(G^\sigma) = S(G)$ for any orientation G^σ of G . Conversely, we assume that $S_p(G^\sigma) = S(G)$ for any orientation G^σ of G . From Theorem 4.1, we know that G has no even cycles. By Theorem 3.2, $a_k(G^\sigma) = 0$ for all odd k . Since $S_p(G^\sigma) = S(G)$, we have $b_k(G) = 0$ for all odd k , where $b_k(G)$ is the coefficient of x^{n-k} in the characteristic polynomial of G . It implies that G is bipartite. Thus G is a forest. \square

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